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SECOND-ORDER MULTIOBJECTIVE SYMMETRIC PROGRAMMING PROBLEM AND DUALITY RELATIONS UNDER (F, G_f)-CONVEXITY

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ABSTRACT

In this paper, we introduce the definition (F, G_f)-convexity and give a nontrivial example existing such type of functions. Further, a generalization of convexity, namely (F, G_f)-convexity, is introduced in the case of non-linear multiobjective programming problems where the functions constituting vector optimization problems are differentiable. Further, second-order Wolfe type primal-dual pair has been formulated and proved weak, strong and converse duality theorems under (F, G_f)-convexity assumptions.

Keywords: *Symmetric duality, Non-linear programming,* (F, G_f)*-convexity; Wolfe type model*

I. INTRODUCTION

An Optimization problem is the problem of finding the best solution from all feasible solutions. Generally, all problems to be optimized should be able to be formulated as a system with its status controlled by one or more input variables and its performance specified by a well defined Objective Function.The goal of optimization is to find the best value for each variable in order to achieve satisfactory performance.Optimization is an active and fast growing research area and has a great impact on the real world.

One practical advantage of second order duality is that it provides tighter bounds for the value of the objective function of the primal problem when approximations are used because there are more parameters involved. Duality is one of the most important topic in operation research as it helps us in generating useful insights about the optimization problem. Mangasarian [11] is introduced the concept of second-order duality for nonlinear programming . Two distinct pairs of second-order symmetric dual problems under generalized bonvexity/ boncavity assumptions studied in Gulati et al. [7]. Furthermore, Gulati and Gupta [8] have been introduced the concept of η_1-bonvexity/ η_2-boncavity and derived duality results for a Wolfe type model.

The concept of G-invex function has been introduced by Antczak [1] and further derived some optimality conditions for constrained optimization problem. Extended the above notion by defining a vector valued G_f- invex function, Antzcak [2] proved necessary and sufficient optimality conditions for a multiobjective nonlinear programming problem. In last several years, various optimality and duality results have been obtained for multiobjective fractional programming problems. In Chen [3], multiobjective fractional problem and its duality theorems have been considered under higher-order (F, α, ρ, d)- convexity. Later on, Suneja et al. [13] discussed higher-order Mond-Weir and Schaible type nondifferentiable dual programs and their duality theorems under higher-order (F, ρ, σ) -type I-assumptions. Recently, several researchers like (see[5, 6, 14]) have also worked in the same direction.

In this article, we have introduce the definition of (F, G_f)-convex function. Further, we construct a nontrivial numerical example which is (F, G_f)-convex but it is neither second-order F-convex nor F-convex. Also, we have consider Wolfe type multiobjective second-order symmetric dual program and establish the duality relations under

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(F,G_f)- convexity assumptions.

II. NOTATIONS AND PRELIMINARIES

Consider the following vector minimization problem:

(MP) $$\text{Minimize } f(x) = \left\{ f_1(x), f_2(x), ..., f_k(x) \right\}^T$$

$$\text{Subject to } X^0 = \{x \in X \subset R^n : g_j(x) \le 0,\ j = 1,2,...,m\}$$

where $f = \{f_1, f_2, ..., f_k\} : X \to R^k$ and $g = \{g_1, g_2, ..., g_m\} : X \to R^m$ are differentiable functions defined on X.

Definition 2.1 A point $\bar{x} \in X^0$ is said to be an efficient solution of (MP) if there exists no other $x \in X^0$ such that $f_r(x) < f_r(\bar{x})$, for some $r = 1,2,...,k$ and $f_i(x) \le f_i(\bar{x})$, for all $i = 1,2,...,k$.

Definition 2.2. A functional $F : X \times X \times R^n \to R$ is said to be sublinear with respect to the third variable if for all $(x,u) \in X \times X$,

(i) $F_{x,u}(a_1 + a_2) \le F_{x,u}(a_1) + F_{x,u}(a_2)$, for all $a_1, a_2 \in R^n$,
(ii) $F_{x,u}(\alpha a) = \alpha F_{x,u}(a)$, for all $\alpha \in R_+$ and $a \in R^n$.

Many generalizations of the definition of a convex function have been introduced in optimization theory in order to weak the assumption of convexity for establishing optimality and duality results for new classes of nonconvex optimization problems, including vector optimization problems. One of such a generalization of convexity in the vectorial case is the G-invexity notion introduced by Antczak for differentiable scalar and vector optimization problems (see [1, 2] respectively). We now generalize and extend it to the nondifferentiable vectorial case namely, motivated by Jeyakumar and Mond [12] and Antczak [2], we introduce the concept of (F,G_f)-convex.

Definition 2.3. Let $f : X \to R^k$ be a vector-valued differentiable function. If there exist a sublinear functional F and differentiable function $G_f = (G_{f_1}, G_{f_2}, ..., G_{f_k}) : R \to R^k$ such that every component $G_{f_i} : I_{f_i}(X) \to R$ is strictly increasing on the range of I_{f_i} such that $\forall\, x \in X$ and $p \in R^n$,

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) \ \ge F_{x,u}[G'_{f_i}(f_i(u))\nabla_x f_i(u) + \{G''_{f_i}(f_i(u))\nabla_x f_i(u)(\nabla_x f_i(u))^T + G'_{f_i}(f_i(u))\nabla_{xx} f_i(u)\}p]$$
$$- \frac{1}{2}p^T[G''_{f_i}(f_i(u))\nabla_x f_i(u)(\nabla_x f_i(u))^T + G'_{f_i}(f_i(u))\nabla_{xx} f_i(u)]p, \text{ for all } i = 1,2,...,k,$$

then f is called (F,G_f)-convex at $u \in X$.
If the above inequality sign changes to $\le$, then f is called (F,G_f)-concave at $u \in X$ with respect to η.
If the function $f_i, i = 1,2,...,k$ satisfies above inequality, then we will say that f_i is (F,G_{f_i})-convex at $u \in X$.
If the above inequalities sign changes to $\le$, then f is called (F,G_f)-concave at $u \in X$.

Remark 2.1 If $F_{x,u}(a) = \eta^T(x,u)a$, then Definition 2.3 becomes G_f-bonvex given by [9].

Now, we give a nontrivial example which is (F,G_f)-convex function but not F-convex function.

Example 2.1. Let $f : [-1,1] \to R^2$ be defined as

$$f(x) = \{f_1(x), f_2(x)\},$$

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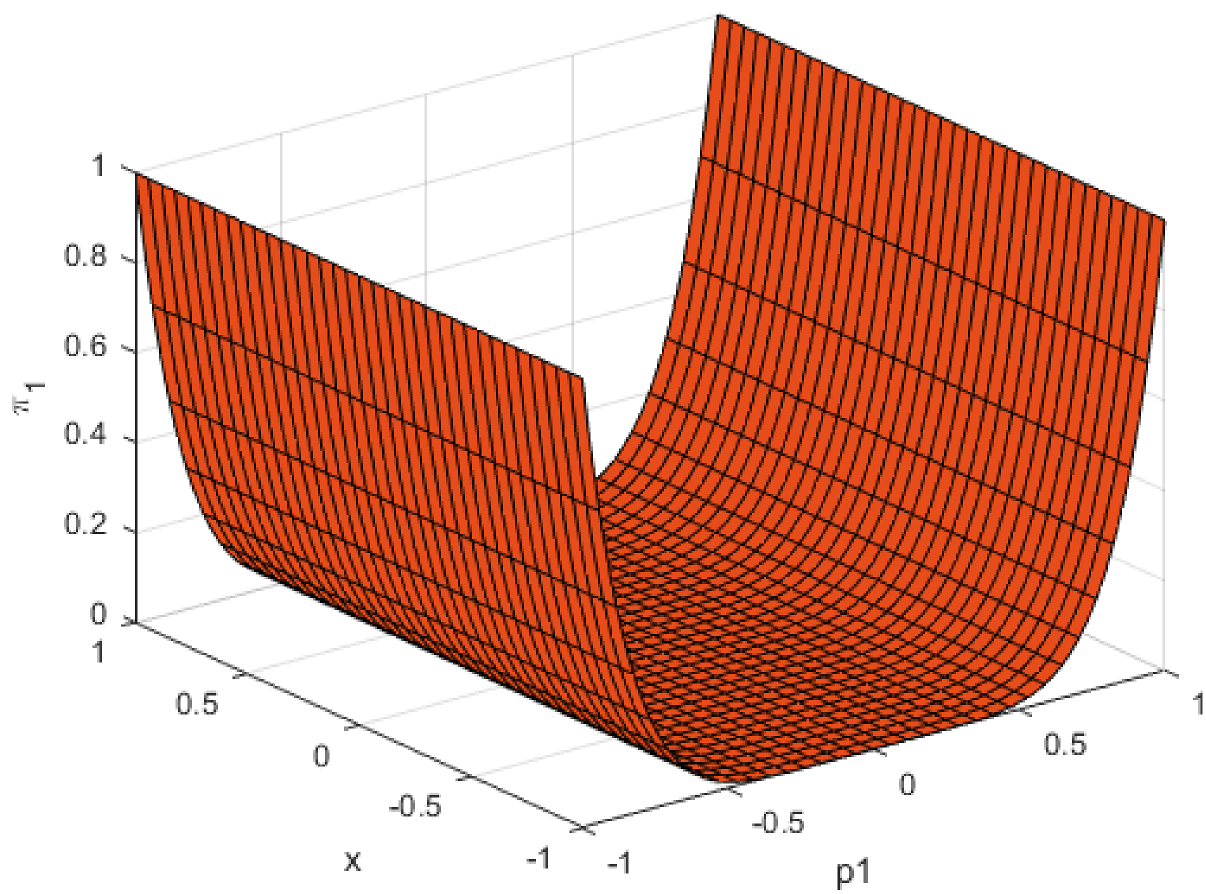

Figure 1:

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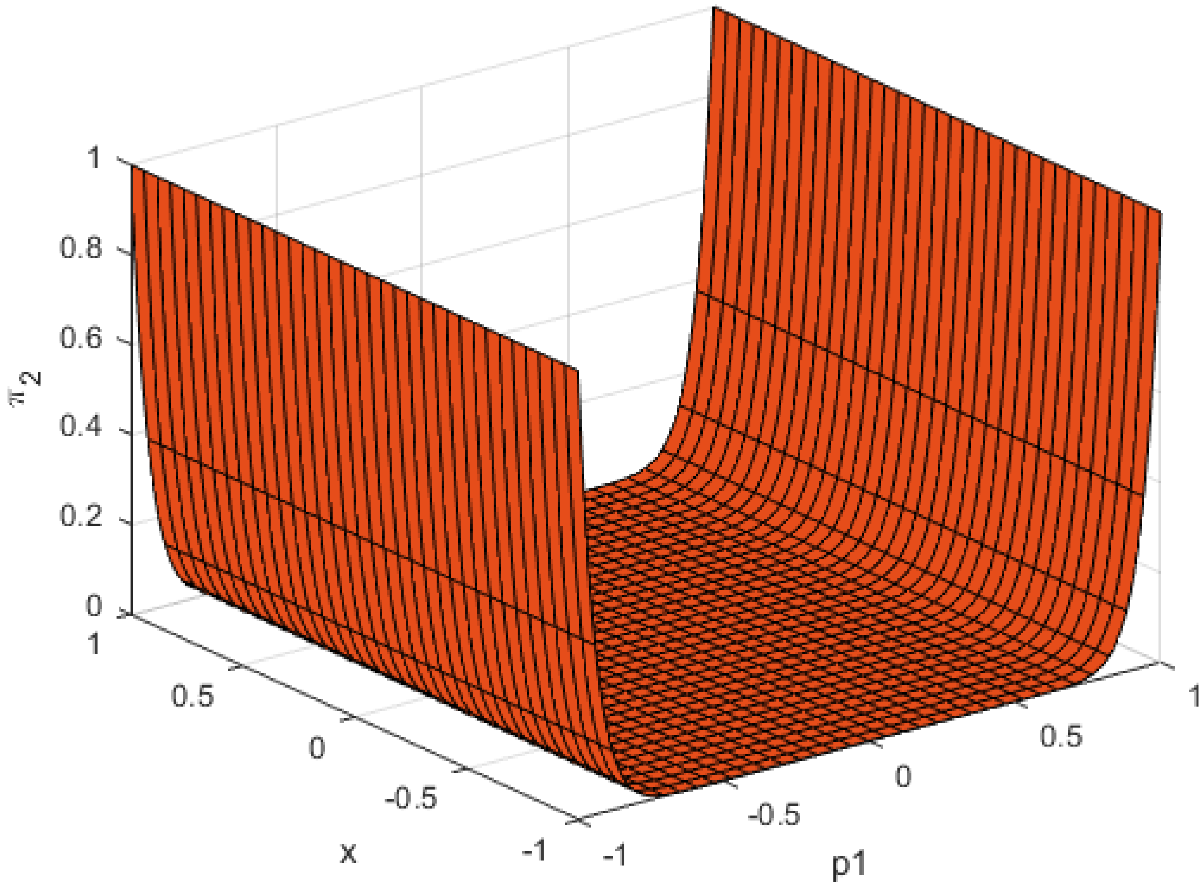

Figure 2:

where $f_1(x) = x^3,\ f_2(x) = x^4$ and $G_f = \{G_{f_1}, G_{f_2}\} : R \to R^2$ be defined as:

$$G_{f_1}(t) = t^2 + 1,\ G_{f_2}(t) = t^4.$$

Let $F : X \times X \times R^2 \to R$ be given as:

$$F_{x,u}(a) = |a|(x^2 - u^2).$$

For showing that f is (F, G_f)-convex at $u = 0$, for this we have to claim that

$$\pi_i = G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) - F_{x,u}[G'_{f_i}(f_i(u))\nabla_x f_i(u) + \{G''_{f_i}(f_i(u))\nabla_x f_i(u)(\nabla_x f_i(u))^T +$$
$$G'_{f_i}(f_i(u))\nabla_{xx} f_i(u)\}p_i] + \frac{1}{2}p_i^T[G''_{f_i}(f_i(u))\nabla_x f_i(u)(\nabla_x f_i(u))^T + G'_{f_i}(f_i(u))\nabla_{xx} f_i(u)]p_i \geq 0, \text{ for } i = 1, 2.$$

Putting the values of $f_1,\ f_2,\ G_{f_1}$ and G_{f_2} in the above expressions, we have

$$\pi_1 = (x^6 + 1) - (u^6 + 1) - F_{x,u}[6u^5 + \{18u^4 + 6u^5\}p_1] + \frac{1}{2}p_1^2\{18u^4 + 6u^5\},$$

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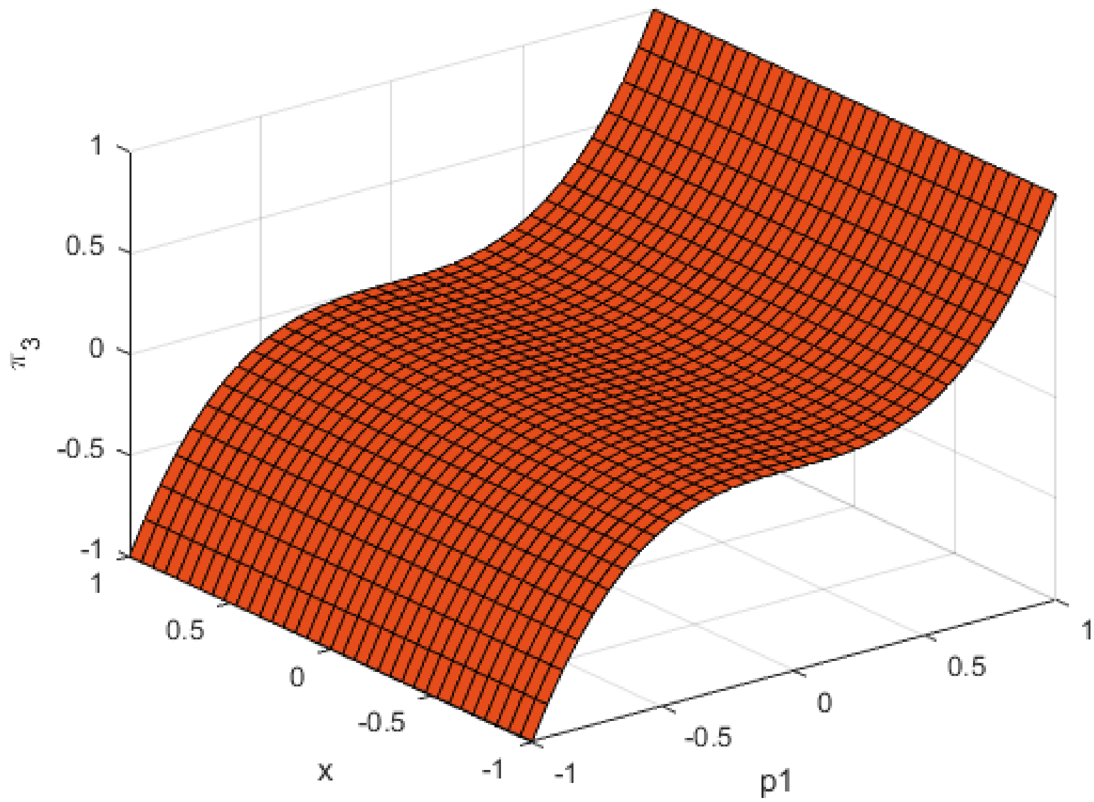

Figure 3:

and

$$\pi_2 = x^{16} - u^{16} - -F_{x,u}(16u^{15} + \{12 \times 16u^{17} + 48u^{14}\}p_2) + \frac{1}{2}p_2^2\{12 \times 16u^{17} + 48u^{14}\},$$

At $u = 0 \in [-1, 1]$, the above expressions reduces:

$$\pi_1 = x^6 \text{ and } \pi_2 = x^{16} \text{ for all } x \in [-1, 1]$$

and hence $\pi_1 \geq 0$ and $\pi_2 \geq 0$, $\left(\text{from figures } (1) \text{ and } (2)\right)$, for all $x \in [-1, 1]$.
Therefore, f is (F, G_f)-convex at $u = 0$.
Now, suppose

$$\pi_3 = f_1(x) - f_1(u) - F_{x,u}[\nabla_x f_1(u) + \nabla_{xx} f_1(u)p_1] + \frac{1}{2}p_1^T[\nabla_{xx} f_1(u)]p_1.$$

or

$$\pi_3 = x^3 - u^3 - F_{x,u}[3u^2 + 6up_1] + 3up_1^2$$

which at $u = 0$ yields

$$\pi_3 = x^3, \text{ for all } x \in [-1, 1].$$

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Obviously,

$$\pi_3 \ngeq 0, \left(\text{from figure (3)}\right).$$

Therefore, f_1 is not second-order F-convex at $u = 0$ with respect to p. Hence, $f = (f_1,\ f_2)$ is not second-order F-convex at $u = 0$ with respect to p.
Finally, consider

$$\xi = f_1(x) - f_1(u) - F_{x,u}(\nabla_x f_1(u))$$

or

$$\xi = x^3 - u^3 - F_{x,u}(3u^2)$$

which at $u = 0$ and using sublinearity of functional F, we get

$$\xi = x^3.$$

At the point $x = \frac{-1}{3} \in [-1,1]$, obtain

$$\xi = \frac{-1}{27} \ngeq 0.$$

Therefore, f_1 is not F-convex at $u = 0$. Hence, $f = (f_1,\ f_2)$ is not F-convex at $u = 0$.

III. WOLFE TYPE SYMMETRIC DUAL PROGRAM

Consider the following pair of Wolfe type dual program:
Primal problem (WP):
Minimize $R(x,y,\lambda,p) = \Big(R_1(x,y,\lambda_1,p), R_2(x,y,\lambda_2,p), ...R_k(x,y,\lambda_k,p)\Big)^T$

Subject to

$$\sum_{i=1}^{k} \lambda_i \Big[G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y) + \{G''_{f_i}(f_i(x,y))\nabla_y f_i(x,y)(\nabla_y f_i(x,y))^T + G'_{f_i}(f_i(x,y))\nabla_{yy} f_i(x,y)\}p\Big] \leq 0, \tag{1}$$

$$\lambda > 0,\ \ \lambda^T e_k = 1. \tag{2}$$

Dual problem (WD):
Maximize $S(u,v,\lambda,q) = \Big(S_1(u,v,\lambda_1,q), S_2(u,v,\lambda_2,q), ..., S_k(u,v,\lambda_k,q)\Big)^T$

Subject to

$$\sum_{i=1}^{k} \lambda_i \Big[G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v) + \{G''_{f_i}(f_i(u,v))\nabla_x f_i(u,v)(\nabla_x f_i(u,v))^T + G'_{f_i}(f_i(u,v))\nabla_{xx} f_i(u,v))\}q\Big] \geq 0, \tag{3}$$

$$\lambda > 0,\ \ \lambda^T e_k = 1, \tag{4}$$

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where for all $i = 1,2,...,k$,

$$R_i(x,y,\lambda,p) = G_{f_i}(f_i(x,y)) - y^T \sum_{i=1}^{k} \lambda_i \Big(G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y) + [G''_{f_i}(f_i(x,y))\nabla_y f_i(x,y)(\nabla_y f_i(x,y))^T G'_{f_i}(f_i(x,y))\nabla_{yy} f_i(x,y)]p \Big)$$

$$-\frac{1}{2}\sum_{i=1}^{k} \lambda_i p^T \Big(G''_{f_i}(f_i(x,y))\nabla_y f_i(x,y)(\nabla_y f_i(x,y))^T + G'_{f_i}(f_i(x,y))\nabla_{yy} f_i(x,y) \Big) p,$$

$$S_i(u,v,\lambda,q) = G_{f_i}(f_i(u,v)) - u^T \sum_{i=1}^{k} \lambda_i \Big(G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v) + [G''_{f_i}(f_i(u,v))\nabla_x f_i(u,v)(\nabla_x f_i(u,v))^T + G'_{f_i}(f_i(u,v))\nabla_{xx} f_i(u,v)]q \Big)$$

$$-\frac{1}{2}\sum_{i=1}^{k} \lambda_i q^T \Big(G''_{f_i}(f_i(u,v))\nabla_x f_i(u,v)(\nabla_x f_i(u,v))^T + G'_{f_i}(f_i(u,v))\nabla_{xx} f_i(u,v) \Big) q,$$

(i) f_i and G_{f_i} are differentiable functions in x and y, $e_k = (1,1,...,1)^T \in R^k$,
(ii) p_i and q_i are vectors in R^m and R^n, respectively, $\lambda \in R^k$.

The following example shows the feasibility of the (WP) and (WD) problem discussed above:

Example 3.1 Let $k = 2$. Let $f_i : X \times Y \to R$ be defined as

$$f_1(x,y) = x^3,\ f_2(x,y) = y^3.$$

Suppose $G_{f_i}(t) = t,\ i = 1,2.$

(EWP) Minimize
$R(x,y) = (x^3 - y\lambda_2[3y^2 + 6yp] - 3\lambda_2 yp^2,\ y^3 - y\lambda_2[3y^2 + 6yp] - 3\lambda_2 yp^2)$

Subject to

$$\lambda_2(3y^2 + 6yp) \leq 0,$$

$$\lambda_1, \lambda_2 > 0, \lambda_1 + \lambda_2 = 1.$$

(EWD) Maximize

$S(u,v) = (u^3 - u\lambda_1[3u^2 + 6uq] - 3\lambda_1 q^2 u,\ v^3 - u\lambda_1[3u^2 + 6uq] - 3\lambda_1 q^2 u)$

Subject to

$$\lambda_1[3u^2 + 6uq] \geq 0,$$

$$\lambda_1, \lambda_2 > 0, \lambda_1 + \lambda_2 = 1.$$

One can easily verify that $x = 3, y = 0, \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{2}, p = 2$ is a feasible solution of primal problem and $u = 0, v = \frac{1}{3}, \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{2}, q = 1$ is a feasible solution of dual problem. This shows that such primal-dual pair (EWP) and (EWD) exist.

Next, we prove duality theorems for the pair (WP) and (WD).

Theorem 3.1 (Weak duality theorem). *Let* (x,y,λ,p) *and* (u,v,λ,q) *be feasible solutions of primal and dual problem, respectively. Let for all* $i = 1,2,...,k$,

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(i) $f_i(.,v)$ be (F, G_{f_i})-convex at u ,

(ii) $f_i(x,.)$ be (H, G_{f_i})- concave at y,

where the sublinear functionals $F : R^n \times R^n \times R^n \to R$ and $H : R^m \times R^m \times R^m \to R$ satisfy the following conditions:

(iii) $F_{x,u}(a) + a^T u \geq 0, \; \forall \, a \in R^n_+$,

(iv) $H_{v,y}(b) + b^T y \geq 0, \; \forall \, b \in R^m_+$.

Then, the following cannot hold:

$$R_i(x,y,\lambda,p) \leq S_i(u,v,\lambda,q), \text{ for all } i = 1,2,...,k \tag{5}$$

and

$$R_r(x,y,\lambda,p) < S_r(u,v,\lambda,q), \text{ for some } r = 1,2,...,k. \tag{6}$$

Proof. Suppose on the contrary that (5) and (6) hold. Then, using $\lambda > 0$, we obtain

$$\sum_{i=1}^{k} \lambda_i \Big[G_{f_i}(f_i(x,y)) - y^T \sum_{i=1}^{k} \lambda_i \Big(G'_{f_i}(f_i(x,y)) \nabla_y f_i(x,y) + G''_{f_i}(f_i(x,y)) \nabla_y f_i(x,y) (\nabla_y f_i(x,y))^T + G'_{f_i}(f_i(x,y))$$
$$\nabla_{yy} f_i(x,y)]p \Big) - \frac{1}{2} \sum_{i=1}^{k} \lambda_i \Big(p^T [G''_{f_i}(f_i(x,y)) \nabla_y f_i(x,y) (\nabla_y f_i(x,y))^T + G'_{f_i}(f_i(x,y)) \nabla_{yy} f_i(x,y)]p \Big) \Big]$$
$$< \sum_{i=1}^{k} \lambda_i \Big[G_{f_i}(f_i(u,v)) - u^T \sum_{i=1}^{k} \lambda_i \Big(G'_{f_i}(f_i(u,v)) \nabla_x f_i(u,v) + G''_{f_i}(f_i(u,v)) \nabla_x f_i(u,v) (\nabla_x f_i(u,v))^T + G'_{f_i}(f_i(u,v))$$
$$\nabla_{xx} f_i(u,v)]q \Big) - \frac{1}{2} \sum_{i=1}^{k} \lambda_i \Big(q^T [G''_{f_i}(f_i(u,v)) \nabla_x f_i(u,v) (\nabla_x f_i(u,v))^T + G'_{f_i}(f_i(u,v)) \nabla_{xx} f_i(u,v)]q \Big) \Big]. \tag{7}$$

Since $f_i(.,v)$ is (F, G_{f_i})-convex at u, we get

$$G_{f_i}(f_i(x,v)) - G_{f_i}(f_i(u,v)) \geq F_{x,u}[G'_{f_i}(f_i(u,v)) \nabla_x f_i(u,v) + \{G''_{f_i}(f_i(u,v)) \nabla_x f_i(u,v) (\nabla_x f_i(u,v))^T + G'_{f_i}(f_i(u,v))$$
$$\nabla_{xx} f_i(u,v)\}q] - \frac{1}{2} q^T [G''_{f_i}(f_i(u,v)) \nabla_x f_i(u,v) (\nabla_x f_i(u,v))^T + G'_{f_i}(f_i(u,v)) \nabla_{xx} f_i(u,v)]q.$$

Since $\lambda > 0$ and using sublinearity of F at the third position, the above inequality yields

$$\sum_{i=1}^{k} \lambda_i [G_{f_i}(f_i(x,v)) - G_{f_i}(f_i(u,v))] \geq F_{x,u} \Big\{ \sum_{i=1}^{k} \lambda_i [G'_{f_i}(f_i(u,v)) \nabla_x f_i(u,v) + \{G''_{f_i}(f_i(u,v)) \nabla_x f_i(u,v) (\nabla_x f_i(u,v))^T$$
$$+ G'_{f_i}(f_i(u,v)) \nabla_{xx} f_i(u,v)\}q] \Big\} - \frac{1}{2} \sum_{i=1}^{k} \lambda_i q^T [G''_{f_i}(f_i(u,v)) \nabla_x f_i(u,v) (\nabla_x f_i(u,v))^T + G'_{f_i}(f_i(u,v)) \nabla_{xx} f_i(u,v)]q.$$

Further, using hypothesis (iii) and dual constraint (3), we get

$$\sum_{i=1}^{k} \lambda_i [G_{f_i}(f_i(x,v)) - G_{f_i}(f_i(u,v))] \geq -u^T \sum_{i=1}^{k} \lambda_i [G'_{f_i}(f_i(u,v)) \nabla_x f_i(u,v) + \{G''_{f_i}(f_i(u,v)) \nabla_x f_i(u,v) (\nabla_x f_i(u,v))^T$$
$$+ G'_{f_i}(f_i(u,v)) \nabla_{xx} f_i(u,v)\}q] - \frac{1}{2} \sum_{i=1}^{k} \lambda_i q^T [G''_{f_i}(f_i(u,v)) \nabla_x f_i(u,v) (\nabla_x f_i(u,v))^T + G'_{f_i}(f_i(u,v)) \nabla_{xx} f_i(u,v)]q. \tag{8}$$

Similarly, using hypotheses (i), (iv) and dual constraint (1), we get

$$\sum_{i=1}^{k}\lambda_i\big[-G_{f_i}(f(x,v))+G_{f_i}(f_i(x,y))\big]\geq y^T\sum_{i=1}^{k}\lambda_i\big[G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y)+\{G''_{f_i}(f_i(x,y))\nabla_y f_i(x,y)(\nabla_y f_i(x,y))^T$$

$$+G'_{f_i}(f_i(x,y))\nabla_{yy}f_i(x,y)\}p\big]+\frac{1}{2}\sum_{i=1}^{k}\lambda_i p^T[G''_{f_i}(f_i(x,y))\nabla_y f_i(x,y)(\nabla_y f_i(x,y))^T+G'_{f_i}(f_i(x,y))\nabla_{yy}f_i(x,y)]p. \quad (9)$$

Finally, adding inequalities (8), (9) and $\lambda^T e_k=1$, we get

$$\sum_{i=1}^{k}\lambda_i\Bigg[G_{f_i}(f_i(x,y))-y^T\sum_{i=1}^{k}\lambda_i\bigg(G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y)+G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y)(\nabla_y f_i(x,y))^T+G'_{f_i}(f_i(x,y))$$

$$\nabla_{yy}f_i(x,y)]p\bigg)-\frac{1}{2}\sum_{i=1}^{k}\lambda_i\bigg(p^T[G''_{f_i}(f_i(x,y))\nabla_y f_i(x,y)(\nabla_y f_i(x,y))^T+G'_{f_i}(f_i(x,y))\nabla_{yy}f_i(x,y)]p\bigg)\Bigg]$$

$$\geq\sum_{i=1}^{k}\lambda_i\Bigg[G_{f_i}(f_i(u,v))-u^T\sum_{i=1}^{k}\lambda_i\bigg(G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v)+G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v)(\nabla_x f_i(u,v))^T+G'_{f_i}(f_i(u,v))$$

$$\nabla_{xx}f_i(u,v)]q\bigg)-\frac{1}{2}\sum_{i=1}^{k}\lambda_i\bigg(q^T[G''_{f_i}(f_i(u,v))\nabla_x f_i(u,v)(\nabla_x f_i(u,v))^T+G'_{f_i}(f_i(u,v))\nabla_{xx}f_i(u,v)]q\bigg)\Bigg].$$

This contradicts (7). Hence, the result.

Theorem 3.2 (Strong duality). Let $(\bar{x},\bar{y},\bar{\lambda},\bar{p})$ be an efficient solution of (WP); fix $\lambda=\bar{\lambda}$ in (WD) such that

(i) for all $i=1,2,...,k$, $[G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T+G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y})]$ is nonsingular,

(ii) $\displaystyle\sum_{i=1}^{k}\bar{\lambda}_i\nabla_y\bigg(\{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T+G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y})\}\bar{p}\bigg)\bar{p}$
$\notin \text{span}\Big\{G'_{f_1}(f_1(\bar{x},\bar{y}))\nabla_y f_1(\bar{x},\bar{y}),...,G'_{f_k}(f_k(\bar{x},\bar{y}))\nabla_y f_k(\bar{x},\bar{y})\Big\}\setminus\{0\}$,

(iii) the vectors $\Big\{G'_{f_1}(f_1(\bar{x},\bar{y}))\nabla_y f_1(\bar{x},\bar{y}),G'_{f_2}(f_2(\bar{x},\bar{y}))\nabla_y f_2(\bar{x},\bar{y}),...,G'_{f_k}(f_k(\bar{x},\bar{y}))\nabla_y f_k(\bar{x},\bar{y})\Big\}$ are linearly independent,

(iv) $\displaystyle\sum_{i=1}^{k}\bar{\lambda}_i\nabla_y\bigg(\{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T+G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y})\}\bar{p}\bigg)\bar{p}=0\Rightarrow\bar{p}=0.$

Then, $\bar{q}=0$ such that $(\bar{x},\bar{y},\bar{\lambda},\bar{q}=0)$ is feasible solution for (WD) and the value of objective functions are equal. Also, if the assumptions of weak duality theorem hold, then $(\bar{x},\bar{y},\bar{\lambda},\bar{q}=0)$ is an efficient solution for (WD).

Proof. Since $(\bar{x},\bar{y},\bar{\lambda},\bar{p})$ is an efficient solution of (WP), using Fritz -John necessary conditions [4], then there exist $\alpha\in R^k$, $\beta\in R^m$ and $\eta\in R$ such that

$$\sum_{i=1}^{k}\alpha_i[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y})]+\sum_{i=1}^{k}\bar{\lambda}_i\Big[G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})\nabla_x f_i(\bar{x},\bar{y})+G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{xy}f_i(\bar{x},\bar{y})\Big]\Big(\beta-(\alpha^T e_k)\bar{y}\Big)$$

$$+\sum\bar{\lambda}_i\nabla_x\Big[(G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T+G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y}))\bar{p}\Big]\Big(\beta-(\alpha^T e_k)(\bar{y}+\frac{1}{2}\bar{p})\Big)=0, \quad (10)$$

$$\sum_{i=1}^{k}\left(\alpha_i-(\alpha^T e_k)\bar{\lambda}_i\right)\left[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})\right]+\sum_{i=1}^{k}\bar{\lambda}_i\left[\left\{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T\right.\right.$$
$$\left.+G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y})\right\}\left(\beta-(\alpha^T e_k)(\bar{y}+\bar{p})\right)+\sum_{i=1}^{k}\bar{\lambda}_i\left[\nabla_y\{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T\right.$$
$$\left.\left.+G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y}))\bar{p}\}\right]\left(\beta-(\alpha^T e_k)(\bar{y}+\frac{1}{2}\bar{p})\right)\right]=0, \tag{11}$$

$$\left[G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T+G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y})\right]\left[\left(\beta-(\alpha^T e_k)(\bar{p}+\bar{y})\right)\bar{\lambda}_i\right]=0,\ i=1,2,...,k, \tag{12}$$

$$G'_f(f(\bar{x},\bar{y}))\nabla_y f(\bar{x},\bar{y})\left(\beta-(\alpha^T e_k\bar{y})\right)+\eta e_k+\left\{\left(\beta-(\alpha^T e_k)(\bar{y}+\tfrac{1}{2}\bar{p}_1)\right)^T\left(G''_{f_1}(f_1(\bar{x},\bar{y}))\nabla_y f_1(\bar{x},\bar{y})(\nabla_y f_1(\bar{x},\bar{y}))^T\right.\right.$$
$$\left.+G'_{f_1}(f_1(\bar{x},\bar{y}))\nabla_{yy}f_1(\bar{x},\bar{y}))\bar{p}_1\right),...,\left(\beta-(\alpha^T e_k)(\bar{y}+\tfrac{1}{2}\bar{p}_k)\right)^T\left(G''_{f_k}(f_k(\bar{x},\bar{y}))\nabla_y f_k(\bar{x},\bar{y})(\nabla_y f_k(\bar{x},\bar{y}))^T\right.$$
$$\left.\left.+G'_{f_k}(f_k(\bar{x},\bar{y}))\nabla_{yy}f_k(\bar{x},\bar{y}))\bar{p}_k\right)\right\}=0, \tag{13}$$

$$\beta^T\sum_{i=1}^{k}\bar{\lambda}_i[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})+\{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T+G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y})\}\bar{p}]=0, \tag{14}$$

$$\eta^T[\bar{\lambda}^T e_k-1]=0, \tag{15}$$

$$(\alpha,\beta)\geq 0,\ \ (\alpha,\beta,\eta)\neq 0. \tag{16}$$

Equation (13) can be written as

$$G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})\left(\beta-(\alpha^T e_k)\bar{y}\right)+\left(\beta-(\alpha^T e_k)(\bar{y}+\tfrac{1}{2}\bar{p})\right)^T\left((G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T\right.$$
$$\left.+G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y}))\bar{p}\right)+\eta=0,\ i=1,2,...,k. \tag{17}$$

By hypothesis (i) and $\bar{\lambda}_i>0$, for $i=1,2,...,k$, (12) gives

$$\beta=(\alpha^T e_k)(\bar{p}+\bar{y}),\ i=1,2,...,k. \tag{18}$$

If $\alpha=0$, then (18) implies that $\beta=0$. Further, equation (17) gives $\eta=0$. Consequently, $(\alpha,\beta,\eta)=0$, which contradicts (16). Hence, $\alpha\neq 0$, or $\alpha^T e_k>0$.

Using (18) and $\alpha^T e_k>0$ in (11), we get

$$\sum_{i=1}^{k}\bar{\lambda}_i\left[\left(\nabla_y\left\{(G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T+G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y}))\bar{p}\right\}\bar{p}\right)\right]$$
$$=-\frac{2}{\alpha^T e_k}\sum_{i=1}^{k}[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})]\left(\alpha_i-(\alpha^T e_k)\bar{\lambda}_i\right). \tag{19}$$

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It follows from hypothesis (ii) that

$$\sum_{i=1}^{k}\bar{\lambda}_i\left[\left(\nabla_y\left\{(G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T + G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy} f_i(\bar{x},\bar{y}))\bar{p}\right\}\bar{p}\right)\right] = 0. \tag{20}$$

Hence, by hypothesis (iv), we obtain

$$\bar{p} = 0. \tag{21}$$

Therefore, the inequality (18) implies

$$\beta = (\alpha^T e_k)\bar{y}. \tag{22}$$

Now, using (21) in (19), we obtain

$$\sum_{i=1}^{k}\left(\alpha_i - (\alpha^T e_k)\bar{\lambda}_i\right)[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})] = 0.$$

From hypothesis (iii), it yields

$$\alpha_i = (\alpha^T e_k)\bar{\lambda}_i,\;\; i = 1,2,...,k. \tag{23}$$

Using $\alpha^T e_k > 0$, (21)-(23) in (10), we get

$$\bar{x}^T\sum_{i=1}^{k}\bar{\lambda}_i\left[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y})]\right] = 0.$$

Further, using (14), $\alpha^T e_k > 0$, (21) and (22), we have

$$\bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i\left[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_x f_i(\bar{x},\bar{y})]\right] = 0. \tag{24}$$

Hence, $(\bar{x},\bar{y},\bar{\lambda},\bar{q}=0)$ satisfies the constraints (3) and (4) of (WD) and clearly a feasible solution for the dual problem (WD). Hence, the result.

Theorem 3.3 (Converse duality). *Let $(\bar{u},\bar{v},\bar{\lambda},\bar{q})$ be an efficient solution of (WD); fix $\lambda = \bar{\lambda}$ in (WP) such that*

(i) for all $i = 1,2,...,k,$ $[G''_{f_i}(f_i(\bar{u},\bar{v}))\nabla_x f_i(\bar{u},\bar{v})(\nabla_x f_i(\bar{u},\bar{v}))^T + G'_{f_i}(f_i(\bar{u},\bar{v}))\nabla_{xx} f_i(\bar{u},\bar{v})]$ is nonsingular,

(ii) $$\sum_{i=1}^{k}\bar{\lambda}_i\nabla_x\left(\{G''_{f_i}(f_i(\bar{u},\bar{v}))\nabla_x f_i(\bar{u},\bar{v})(\nabla_x f_i(\bar{u},\bar{v}))^T + G'_{f_i}(f_i(\bar{u},\bar{v}))\nabla_{xx} f_i(\bar{u},\bar{v})\}\bar{q}\right)\bar{q}$$
$$\notin \text{span}\left\{G'_{f_1}(f_1(\bar{u},\bar{v}))\nabla_x f_1(\bar{u},\bar{v}),...,G'_{f_k}(f_k(\bar{u},\bar{v}))\nabla_x f_k(\bar{u},\bar{v})\right\}\setminus\{0\},$$

(iii) the vectors $\left\{G'_{f_1}(f_1(\bar{u},\bar{v}))\nabla_x f_1(\bar{u},\bar{v}), G'_{f_2}(f_2(\bar{u},\bar{v}))\nabla_x f_2(\bar{u},\bar{v}),...,G'_{f_k}(f_k(\bar{u},\bar{v}))\nabla_x f_k(\bar{u},\bar{v})\right\}$ are linearly independent,

(iv) $$\sum_{i=1}^{k}\bar{\lambda}_i\nabla_x\left(\{G''_{f_i}(f_i(\bar{u},\bar{v}))\nabla_x f_i(\bar{u},\bar{v})(\nabla_x f_i(\bar{u},\bar{v}))^T + G'_{f_i}(f_i(\bar{u},\bar{v}))\nabla_{xx} f_i(\bar{u},\bar{v})\}\bar{q}\right)\bar{q} = 0 \Rightarrow \bar{q} = 0.$$

Then, $\bar{p}=0$ such that $(\bar{u},\bar{v},\bar{\lambda},\bar{p}=0)$ is feasible solution of (WP) and $R(\bar{u},\bar{v},\bar{\lambda},\bar{p}) = S(\bar{u},\bar{v},\bar{\lambda},\bar{p})$. Also, if the hypotheses of Theorem 3.1 hold, then $(\bar{u},\bar{v},\bar{\lambda},\bar{p}=0)$ is an efficient solution for (WP).

Proof. It follows on the lines of Theorem 3.2.

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